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NOTE ON AXIAL-SHEAR AND CONTOUR VIBRATIONS OF PRISMS

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Abstract—Exact solutions of the equations of the linear theory of elasticity are given for axial-shear modes of vibration of an isotropic, prismatic bar whose normal section is an equilateral triangle or has the equilateral triangle as a module. A family of contour modes is also described for bars with a rhombic section formed of two equilateral triangles and with sections having the rhombus as a module.

INTRODUCTION

Axial-shear and contour modes of vibration of a cylindrical or prismatic bar are those in which the displacements are parallel and perpendicular, respectively, to the generators of the bar and unvarying along its length. The associated frequencies are the long wave limits of certain of the upper branches of the dispersion relation for waves along the bar. Accordingly, exact solutions of the equations of elasticity for such modes are of interest as they supply data for adjusting or testing various approximate equations or solutions. Thus, the high frequency range of Timoshenko's one-dimensional equations of flexural vibrations of bars is improved by employing the exact frequencies of the fundamental axial-shear modes to calculate shear-correction factors[1]. A higher axial-shear mode and a contour mode play similar roles in the Bleustein-Stanley one-dimensional equations of torsional vibrations[2]. Approximate solutions of the three-dimensional equations, for example the finite element solutions by Talbot and Przemieniecki[3], can also profit by comparison of the computed limiting frequencies of the upper extensional, flexural and torsional modes with the exact results for the corresponding axial-shear and contour modes.

Recent applications[3, 4] of approximate methods to the problem of vibrations of an isotropic, elastic prism with an equilateral triangular cross-section motivated a quest for the exact solutions for the axial-shear modes. Those modes are governed by the same differential equation (the two-dimensional wave equation) and boundary conditions (vanishing normal derivative) as are the motions of water waves in a shallow basin or box[5] and the two-dimensional acoustic vibrations of a gas in a cylindrical or prismatic chamber[6]. Both of these phenomena have been subjects of investigation for almost one hundred and fifty years; so it seems unlikely that simple solutions like those for the equilateral triangle should not have been discovered long ago. However, no published solutions appear to be available.

For the special case of a prism with Poisson's ration 1/4 and a rhombic normal section formed of two equilateral triangles, a solution is given for a family of contour modes—modes with displacements normal to the generators of the prism and uniform along its length. Whereas the axial-shear modes are equivoluminal, the contour modes in the rhombic prism are composed of coupled equivoluminal and dilatational waves. The period of the lowest mode of the family was given in a previous paper[7].

The solutions for both the triangular and rhombic sections satisfy traction-free conditions on all planes parallel to the faces of the prisms and regularly spaced at intervals of the altitude of the triangles. Accordingly, a great variety of vibrating prisms with traction-free faces can be formed by juxtaposition of prisms with the triangular or rhombic section. Examples are exhibited.

EQUILATERAL TRIANGULAR SECTION

The simplicity of the solutions for axial-shear vibrations of a prism with an equilateral triangular section stems from the fact that a straight-crested, sinusoidal, shear wave, with its displacement parallel to the generators of the prism and its wave normal parallel or perpendicular to a side of the triangle, reflects from the other sides in such a way that the wave returns upon itself: in the original direction in the case of the "parallel" wave and in the opposite direction in the case of the "perpendicular" wave, as indicated in Fig. 1. The paths of parallel rays, in each case, have the same length for a full circuit and a pair of waves, traveling in opposite directions, forms a steady vibration.

Thus, in a prism with its generators parallel to the axis of z, we are led to consider waves with displacements only axial and of the forms

$$w = A \sin \gamma \, (y \sin \alpha \pm x \cos \alpha \pm ct \pm \epsilon), \tag{1}$$

where c is the velocity of shear waves, $\epsilon = 0$ or $\pi/2$, and α is the angle the wave normal makes with the axis of x. As the components of displacement, u and v, in the plane of the normal section are zero and the third component is independent of z, the components of stress are

$$T_{xx} = T_{yy} = T_{zz} = T_{xy} = 0, \quad T_{yz} = \mu \partial w/\partial y, \quad T_{zx} = \mu \partial w/\partial x, \tag{2}$$

where μ is the shear modulus. Then the displacement equations of motion, in linear, isotropic elasticity, reduce to

$$c^2(\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2) = \partial^2 w/\partial t^2. \tag{3}$$

The boundary of the normal section is given by the equation

$$x(x - \sqrt{3}y + h)(x + \sqrt{3}y + h) = 0 \tag{4}$$

i.e. an equilateral triangle with altitude h and sides $2h/\sqrt{3}$. From (2) and (4), the conditions for traction-free lateral surfaces of the prism are

$$\begin{aligned} \partial w/\partial x &= 0 \quad \text{on} \quad x = 0, \\ \partial w/\partial x - \sqrt{3}\,\partial w/\partial y &= 0 \quad \text{on} \quad x = \sqrt{3}y - h, \\ \partial w/\partial x + \sqrt{3}\,\partial w/\partial y &= 0 \quad \text{on} \quad x = -\sqrt{3}y - h. \end{aligned} \tag{5}$$

Upon expanding (1) and combining members of the resulting sum for various combinations of the plus and minus signs and values of ϵ and α, we find three independent forms descriptive of waves with wave normals parallel or perpendicular to the sides of the triangle.

For the waves parallel to the sides, $\alpha = \pi/2, \pm\pi/6$ and we have

$$w = (A_1 \cos \gamma y + A_2 \cos \tfrac{1}{2}\sqrt{3}\gamma x \cos \tfrac{1}{2}\gamma y) e^{i\omega t}, \tag{6}$$

$$w = (B_1 \sin \gamma y + B_2 \cos \tfrac{1}{2}\sqrt{3}\gamma x \sin \tfrac{1}{2}\gamma y) e^{i\omega t}. \tag{7}$$

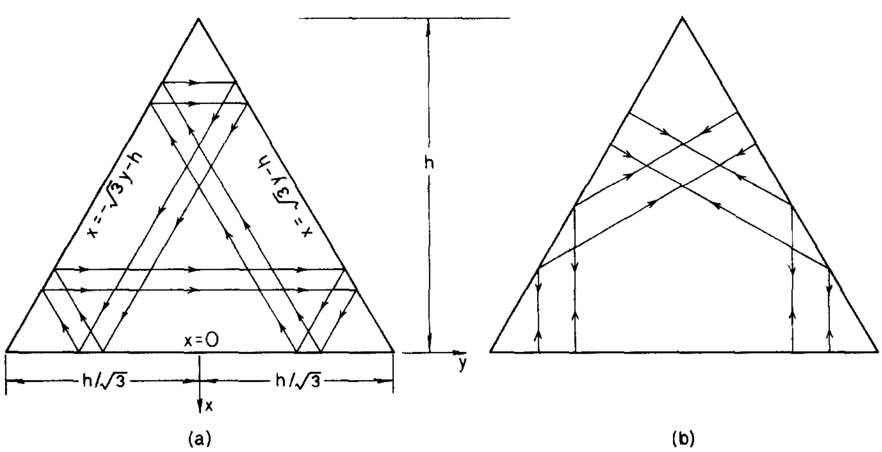

Fig. 1. Illustrating reflections of axial shear waves with wave normals (a) parallel and (b) perpendicular to the faces of a prism with an equilateral triangular normal section.

For the waves perpendicular to the sides, $\alpha = 0, \pm\pi/3$ and we have only

$$w = (C_1 \cos \gamma y + C_2 \cos \tfrac{1}{2}\gamma x \cos \tfrac{1}{2}\sqrt{3}\gamma y)e^{i\omega t} \tag{8}$$

as there can be no companion solution odd in y with straight-crested waves.

Substitution of (6), (7) and (8) in the differential eqn (3) and the boundary conditions (5) yields

Solution A (waves parallel to sides, displacement even in y):

$$w = A[\cos \gamma y + 2(-1)^n \cos \tfrac{1}{2}\sqrt{3}\gamma x \cos \tfrac{1}{2}\gamma y]e^{i\omega t}, \tag{9}$$
$$\omega = \gamma c = 2n\pi c/\sqrt{3}h, \quad n = 1, 2, 3 \ldots$$

Solution B (waves parallel to sides, displacement odd in y):

$$\begin{aligned} w &= B[\sin \gamma y - 2(-1)^n \cos \tfrac{1}{2}\sqrt{3}\gamma x \sin \tfrac{1}{2}\gamma y]e^{i\omega t}, \\ \omega &= \gamma c = 2n\pi c/\sqrt{3}h, \quad n = 1, 2, 3 \ldots \end{aligned} \tag{10}$$

Solution C (waves perpendicular to sides):

$$w = C[\cos \gamma x + 2(-1)^n \cos \tfrac{1}{2}\gamma x \cos \tfrac{1}{2}\sqrt{3}\gamma y]e^{i\omega t}, \tag{11}$$
$$\omega = \gamma c = 2n\pi c/h, \quad n = 1, 2, 3 \ldots$$

The shapes of the first two modes of solutions A and C are illustrated in Figs. 2 and 4. In Fig. 3, for solution B, the third mode is included as it is the lowest axial-shear mode associated with torsional vibrations of the prism; the frequency of the mode $n = 3$ is the exact "warping cut-off frequency" required for the Bleustein-Stanley equations [2].

The frequencies found here do not conform with limiting frequencies of branches shown in Fig. 13 of [3], but the authors agree that the scale of the figure should be reduced by a factor of 2·54/2 (one inch to two centimeters).

SECTIONS WITH EQUILATERAL TRIANGLE AS MODULE

In addition to satisfying the boundary conditions (5), the solutions (9)–(11) satisfy the conditions of zero traction on planes parallel to the lateral faces of the triangular prism and regularly spaced at intervals of the altitude, h, of the triangle, i.e.

$$\begin{aligned} \partial w/\partial x &= 0 \quad \text{on} \quad x = \pm(p-1)h, & p &= 1, 2, 3, \ldots \\ \partial w/\partial x - \sqrt{3}\, \partial w/\partial y &= 0 \quad \text{on} \quad x = \sqrt{3}y \pm (2q-1)h, & q &= 1, 2, 3, \ldots \\ \partial w/\partial x + \sqrt{3}\, \partial w/\partial y &= 0 \quad \text{on} \quad x = -\sqrt{3}y \pm (2r-1)h, & r &= 1, 2, 3, \ldots \end{aligned} \tag{12}$$

Examples of sections for which these conditions are satisfied, for solutions (9)–(11), are illustrated in Figs. 5–7.

RHOMBIC SECTION

The simplest example of the axial-shear solutions described in the preceding section, beyond the single triangle, is the case of the rhombic section given by the equation

$$(x + \sqrt{3}y + h)(x + \sqrt{3}y - h)(x - \sqrt{3}y + h)(x - \sqrt{3}y - h) = 0, \tag{13}$$

i.e. a pair of juxtaposed equilateral triangles. The paths of the rays are illustrated in Figs. 8(a) and (b). In addition, there is a simple family of contour modes if Poisson's ratio of the material is 1/4. A contour mode is one in which the displacements are perpendicular to the generators of the prism and unvarying along its length. Whereas axial-shear modes involve only equivoluminal waves, the contour modes in the rhombic prism comprise coupled equivoluminal and dilatational

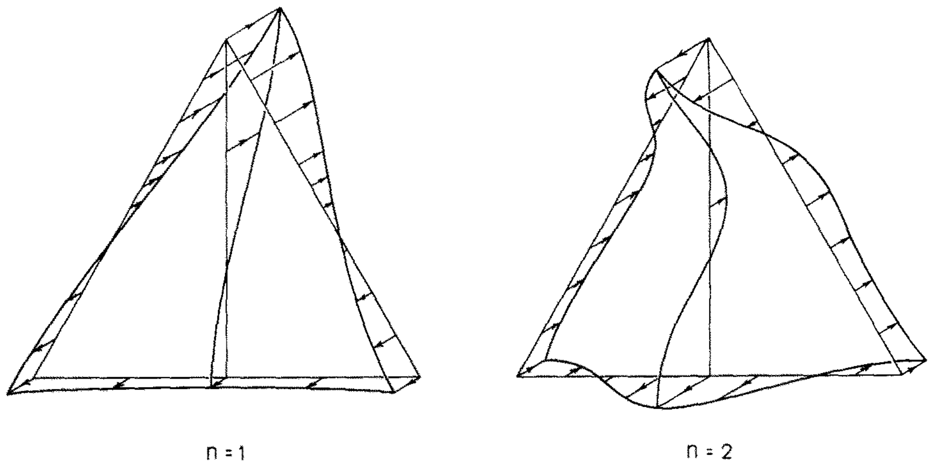

Fig. 2. Axial displacements: Solution A, $n = 1, 2$.

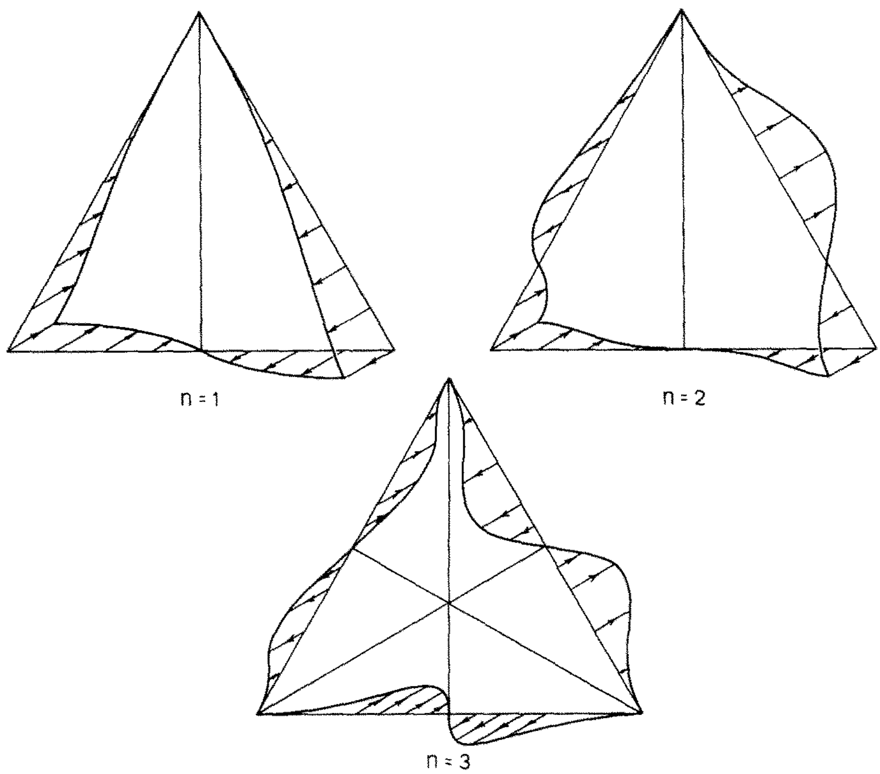

Fig. 3. Axial displacements: Solution B, $n = 1, 2, 3$.

waves. The only known equivoluminal contour modes are the Lamé modes [8] in isotropic prisms with square-module sections and analogous modes [9] in anisotropic prisms with rectangular-module sections.

If Poisson's ratio is 1/4, a dilatational wave, with its normal (and displacement) parallel to the long diagonal of the rhombic section, reflects, on incidence at a boundary, as an equivoluminal wave with its normal parallel (and displacement perpendicular) to the short diagonal. Further, an equivoluminal wave traveling parallel to the short diagonal reflects as a dilatational wave parallel to the long diagonal. Parallel rays form closed rectangular paths of equal length as shown in Fig. 8(c). Pairs of such waves, traveling in opposite directions, form steady vibrations with all displacements parallel to the long diagonal. Accordingly, the components of displacement v and

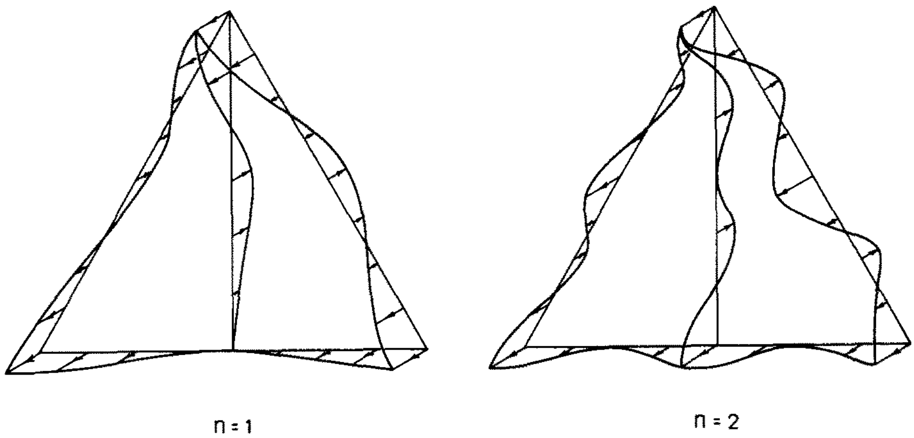

Fig. 4. Axial displacements: Solution C, $n = 1, 2$.

Fig. 5. Examples of normal sections of prisms for which the traction-free boundary conditions (12) are satisfied by the solutions (9)–(11) for axial-shear modes of vibrations.

w, parallel to y and z, are zero and the remaining component has the form

$$u = (D_1 \cos \alpha x + D_2 \cos \beta y) e^{i\omega t}. \tag{14}$$

The components of stress are then

$$\tfrac{1}{3} T_{xx} = T_{yy} = T_{zz} = \mu \partial u / \partial x, \quad T_{yz} = T_{zx} = 0, \quad T_{xy} = \mu \partial u / \partial y \tag{15}$$

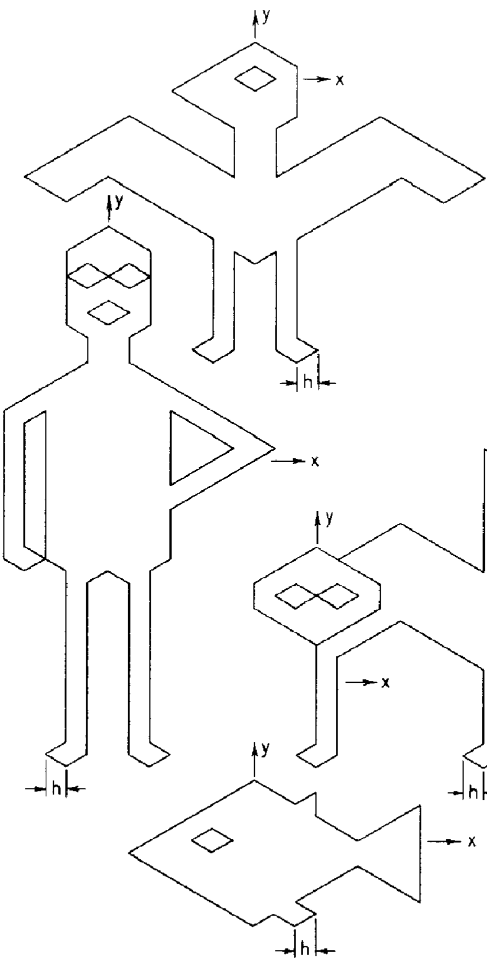

Fig. 6. Examples of normal sections of prisms for which the traction-free boundary conditions (12) are satisfied by the solutions (9)–(11) for axial-shear modes of vibrations.

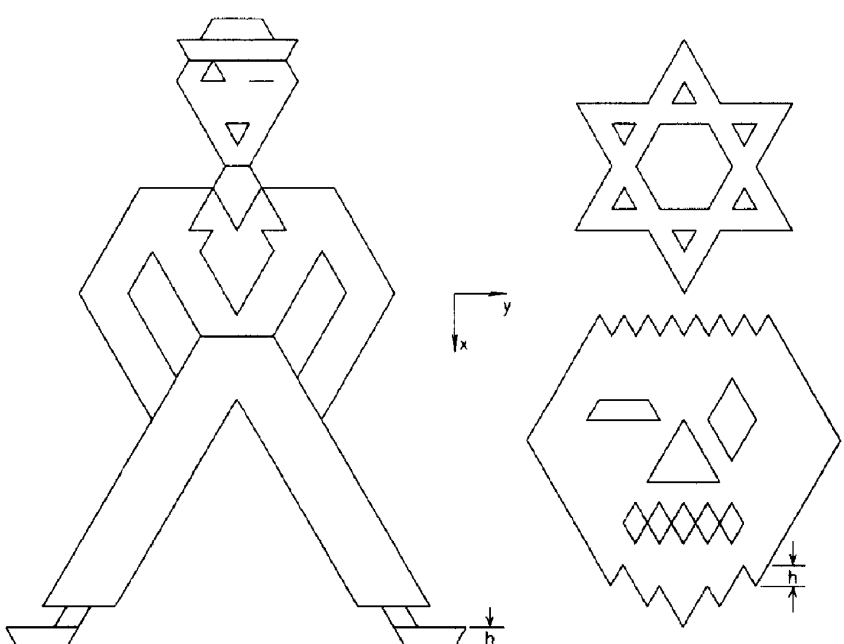

Fig. 7. Examples of normal sections of prisms for which the traction-free boundary conditions (12) are satisfied by the solutions (9)–(11) for axial-shear modes of vibrations.

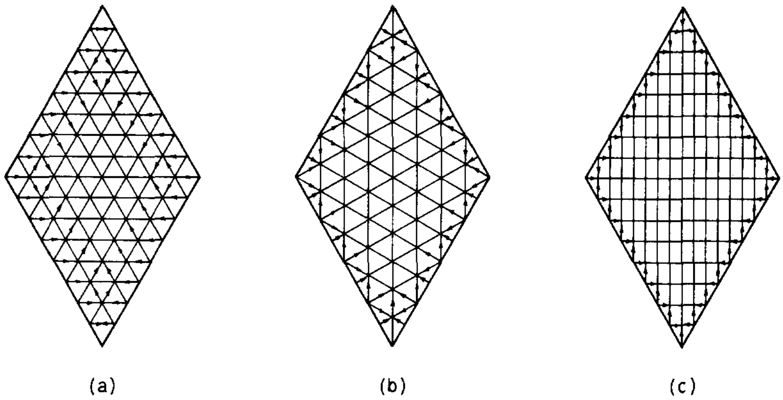

Fig. 8. Illustrating paths of rays in the rhombic section of a prism: in axial-shear modes comprising equivoluminal waves with normals (a) parallel and (b) perpendicular to the faces of the prism; (c) in contour modes comprising coupled equivoluminal and dilatational waves with normals parallel to the short and long diagonals, respectively.

and the displacement equations of motion reduce to

$$c^2(3\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) = \delta^2 u/\delta t^2. \tag{16}$$

Although both tensile and shear stresses contribute, the conditions for traction-free faces of the prism are simply

$$\begin{aligned} \sqrt{3}\,\partial u/\partial x + \partial u/\partial y &= 0 \quad \text{on} \quad x + \sqrt{3}y = \pm h, \\ \sqrt{3}\,\partial u/\partial x - \partial u/\partial y &= 0 \quad \text{on} \quad x - \sqrt{3}y = \pm h, \end{aligned} \tag{17}$$

as the conditions for vanishing normal and tangential traction are the same on each face.

Upon substituting (14) in (16) and (17), we find

$$u = D[\cos \alpha x + (-1)^n \cos \beta y]e^{i\omega t},$$
$$\alpha = \beta/\sqrt{3} = n\pi/h, \quad \omega = \beta c = n\pi c\sqrt{3}/h, \quad n = 1, 2, 3, \ldots \tag{18}$$

The period of the fundamental mode, $n = 1$, in terms of the length of a side, $L = 2h/\sqrt{3}$, is

$$T_1 = \omega_1/2\pi = L/c, \tag{19}$$

a result which was obtained previously [7] simply as the transit time of an equivoluminal wave across the short diagonal or a dilatational wave across the long diagonal.

As in the case of the triangular section, the solution (18), as well as the solutions (9)–(11), produces traction-free planes parallel to the faces of the rhombic prism at intervals h. Thus, a great variety of cross-sectional shapes is also accommodated by the solution (18), although more limited than in the case of the axial shear modes, as there are only two, instead of three, families of traction-free planes.

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